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FLOWS OF AN INCOMPRESSIBLE FLUID WITH LINEAR VORTICITY†

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All the flows of an incompressible fluid, for which the vorticity of the velocity field is linear with respect to the spatial variables, are described.

The condition for the fluid velocity field to be potential field is often used to classify flows in hydrodynamics. However, an arbitrary potential flow of an incompressible fluid is harmonic and is therefore independent of the viscosity. The problem arises of describing classes of flows which generalize potential flows and flows having a non-zero vorticity. This problem is considered below subject to the condition that the vorticity depends linearly on the spatial coordinates.

1. FORMULATION OF THE PROBLEM

We will find all the particular solutions of the equations of motion of an incompressible fluid

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{0},\ \operatorname{div}\mathbf{u} = 0 \tag{1.1}$$

which satisfy the additional condition, or differential relation, $(\operatorname{rot}\mathbf{u})_{ab} = \mathbf{0}$, that is

$$\operatorname{rot}\mathbf{u} = H(t)\mathbf{x} + \mathbf{k}(t) \tag{1.2}$$

In formulae (1.1) and (1.2) and everywhere subsequently, $\mathbf{u} = \{u^a(t, \mathbf{x})\}$ is the fluid velocity field, $p = p(t, \mathbf{x})$ is the pressure, $\mathbf{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\nabla = \{\partial_a\}$, $\Delta = \nabla\cdot\nabla$ is the Laplacian, $H = \{H^{ab}(t)\}$ is a certain 3×3 matrix-function and $\mathbf{k} = \{k^a(t)\}$. The fluid density is assumed to be unity. The coefficient of kinematic viscosity ν is zero in the case of an ideal fluid and is now zero in the case of a viscous fluid. The indices a and b vary from 1 to 3. Summation over repeated indices is implied. The subscripts on the functions denote differentiation with respect to the corresponding variables.

The problem can be reformulated in hydrodynamic terms as follows: it is required to describe all the flows of an incompressible fluid for which the vorticity of the velocity field is linear with respect to the spatial variables. This class of flows include flows with a velocity which is linear or quadratic with respect to $\mathbf{x}$ as subclasses. When $H = 0$ and $\mathbf{k} = \mathbf{0}$, Eq. (1.2) degenerates into the condition for the field $\mathbf{u}$ to be a potential field. On integrating Eq. (1.2), we obtain the local representation for its solution

$$\mathbf{u} = \nabla\varphi + \frac{1}{3}(H\mathbf{x})\times\mathbf{x} + \frac{1}{2}\mathbf{k}\times\mathbf{x} \tag{1.3}$$

where $\varphi = \varphi(t, \mathbf{x})$ is an arbitrary differentiable function.

Representation (1.3) enables one to give a further formulation of this problem: it is required to construct all solutions of Eqs. (1.1) for which the velocity field is a linear superposition of a potential field and a field which is quadratic with respect to the spatial variables.

Remark 1. It is well known [1–5]‡ that the maximum algebra in the Lie sense of the invariance of system (1.1) is the algebra

$$\langle \partial_t, J_{ab}, D^t, D^x, R(\mathbf{m}), Z(\chi)\rangle \tag{1.4}$$

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‡See also the earlier paper: Yu. A. DANILOV, Group properties of Maxwell's and the Navier–Stokes equations, Preprint, I. V. Kurchatov Inst. Atomic Energy, Academy of Sciences of the USSR, Moscow, 1967.

if $\nu = 0$ or the algebra

$$\langle \partial_t, J_{ab}, D^t + \frac{1}{2}D^x, R(\mathbf{m}), Z(\chi) \rangle \tag{1.5}$$

if $\nu \neq 0$. The following notation is used in formulae (1.4) and (1.5)

$$\begin{aligned} &\partial_t = \partial/\partial t, J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial/\partial u^b - u^b\partial/\partial u^a \ (a < b) \\ &D^t = t\partial_t - u^a\partial/\partial u^a - 2p\partial/\partial p, D^x = x_a\partial_a + u^a\partial/\partial u^a + 2p\partial/\partial p \\ &R(\mathbf{m}) = m^a(t)\partial_a + m^a_t(t)\partial/\partial u^a - m^a_{tt}(t)x_a\partial/\partial p \\ &Z(\chi) = \chi(t)\partial/\partial p \end{aligned} \tag{1.6}$$

where $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions (from $C^\infty((t_0, t_1), \mathbf{R})$, for example) and, moreover, this requirement can be substantially relaxed [6].

Remark 2. Operators (1.6) generate the invariant transforms of system (1.1)

$$\partial_t\text{: } \mathbf{u}'(t, \mathbf{x}) = \mathbf{u}(t + \varepsilon, \mathbf{x}), p'(t, \mathbf{x}) = p(t + \varepsilon, \mathbf{x})$$

(a shift with respect to the time t)

$$J_{ab}\text{: } \mathbf{u}'(t, \mathbf{x}) = B\mathbf{u}(t, B^{-1}\mathbf{x}), p'(t, \mathbf{x}) = p(t, B^{-1}\mathbf{x})$$

(spatial rotations; here, B is an arbitrary orthogonal 3×3 matrix)

$$D^t\text{: } \mathbf{u}'(t, \mathbf{x}) = e^\varepsilon\mathbf{u}(e^\varepsilon t, \mathbf{x}), p'(t, \mathbf{x}) = e^{2\varepsilon}p(e^\varepsilon t, \mathbf{x})$$

(scaling with respect to the time t)

$$D^x\text{: } \mathbf{u}'(t, \mathbf{x}) = e^{-\varepsilon}\mathbf{u}(t, e^\varepsilon\mathbf{x}), p'(t, \mathbf{x}) = e^{-2\varepsilon}p(t, e^\varepsilon\mathbf{x})$$

(scaling with respect to the spatial variables)

$$\begin{aligned} &R(\mathbf{m})\text{: } \mathbf{u}'(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{x} - \mathbf{m}(t)) + \mathbf{m}_t(t) \\ &p'(t, \mathbf{x}) = p(t, \mathbf{x} - \mathbf{m}(t)) - \mathbf{m}_{tt} \cdot \mathbf{x} + {}^1/_2\mathbf{m} \cdot \mathbf{m}_{tt} \end{aligned}$$

(transfer to an arbitrary translational moving system of coordinates; these transforms include shifts with respect to spatial variables and a Galilean transformation)

$$Z(\chi)\text{: } \mathbf{u}'(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{x}), p'(t, \mathbf{x}) = p(t, \mathbf{x}) + \chi(t)$$

(pressure changes).

In the case when $\nu \neq 0$, system (1.1) is not invariant with respect to scaling in the time or spatial variables but admits of a grouping of these transforms which is generated by the operator $D^t + {}^1/_2D^x$

$$\mathbf{u}'(t, \mathbf{x}) = e^\varepsilon\mathbf{u}(e^{2\varepsilon}t, e^\varepsilon\mathbf{x}), p'(t, \mathbf{x}) = e^{2\varepsilon}p(e^{2\varepsilon}t, e^\varepsilon\mathbf{x})$$

Remark 3. The invariant transforms of system (1.1) are equivalence transforms [3] for a set of equations of the form of (1.2) if the functions H^{ab} and k^a are assumed to be parameters.

2. PRINCIPAL RESULT

The following theorem describes the structure of the set of solutions of system (1.1), (1.2).

Theorem. Any solution of system (1.1), (1.2), apart from equivalence transforms, locally belongs to one of the following families

1. $H \neq 0$

$$\text{a) } u^1 = (\zeta^1 + \beta^1)x_1 + (\beta^2 - \frac{1}{2}\varkappa)x_2$$

$$u^2 = (\beta^2 + \frac{1}{2}\varkappa)x_1 + (\zeta^1 - \beta^1)x_2$$

$$u^3=-\frac{1}{2}\lambda(x_1^2+x_2^2)+\frac{1}{2}(x_1^2-x_2^2)\mu\sin\theta+x_1x_2\mu\cos\theta-2\zeta^1x_3 \tag{2.1}$$

$$p=-\frac{1}{2}\left(\frac{1}{4}\mu^{-2}(\lambda_t)^2+(\zeta^1)^2+(\zeta^2)^2-\frac{1}{4}(\varkappa)^2+\zeta_t^1\right)(x_1^2+x_2^2)-\frac{1}{2}(\beta_t^1+2\zeta^1\beta^1)(x_1^2-x_2^2)-$$
$$-(\beta_t^2+2\zeta^1\beta^2)x_1x_2+(\zeta_t^1-2(\zeta^1)^2)x_3^2-2\nu\lambda x_3$$

where $\varkappa$, λ, μ, β^1, β^2, ζ^1, ζ^2, θ are smooth functions of the variable t for which the relations

$$\mu>0,\ \mu\mu_t=\lambda\lambda_t,\ \lambda\neq\pm\mu$$
$$\varkappa_t+2\zeta^1\varkappa=0,\ \theta_t=2\lambda\mu^{-1}\zeta^2-\varkappa$$
$$\beta^1=\frac{\lambda_t}{2\mu}\sin\theta+\zeta^2\cos\theta,\ \beta^2=\frac{\lambda_t}{2\mu}\cos\theta-\zeta^2\sin\theta$$

$$\text{b) }\mathbf{u}=-\mu y_1^2\mathbf{e}^3+F^{ab}y_b\mathbf{e}^a+\beta\mathbf{e}^1 \tag{2.2}$$

$$p=\frac{2}{3}\mu(\mathbf{e}_t^3\cdot\mathbf{e}^1)y_1^3-\frac{1}{2}G^{ab}y_ay_b-\left(\beta_t+\left(\zeta-\frac{\mu_t}{2\mu}\right)\beta\right)y_1-2\nu\mu y_3$$

are satisfied, where $y_a=\mathbf{e}^a\cdot\mathbf{x}$, the vectors $\mathbf{e}^a$ form an orthonormal basis which depends smoothly on t, μ, $\varkappa^1$, $\varkappa^2$, ζ, β are smooth functions of the variable t which satisfy the relations

$$\begin{gathered}\mu>0,\ \beta=\mu^{-1}\left(\frac{1}{2}\varkappa^1\varkappa^2-\varkappa^1(\mathbf{e}_t^1\cdot\mathbf{e}^2)-\varkappa^2(\mathbf{e}_t^3\cdot\mathbf{e}^2)\right)\\ \mu_t\mu^{-1}\varkappa^1-2\zeta\varkappa^1+2\varkappa_t^1=0,\ 2(\mathbf{e}_t^3\cdot\mathbf{e}^1)\varkappa^1-2\zeta\varkappa^2-\varkappa_t^2=0\end{gathered} \tag{2.3}$$

F and G are smooth 3×3 matrix functions of the variable t which are defined according to the formulae (δ_{ab} is the Kronecker delta)

$$F=\left\|\begin{matrix}-\frac{1}{2}\mu_t\mu^{-1}+\zeta & \mathbf{e}_t^2\cdot\mathbf{e}^1 & \mathbf{e}_t^3\cdot\mathbf{e}^1\\ \mathbf{e}_t^2\cdot\mathbf{e}^1+\varkappa^2 & \frac{1}{2}\mu_t\mu^{-1}+\zeta & -\mathbf{e}_t^3\cdot\mathbf{e}^2\\ \mathbf{e}_t^3\cdot\mathbf{e}^1 & -\mathbf{e}_t^3\cdot\mathbf{e}^2+\varkappa^1 & -2\zeta\end{matrix}\right\|$$

$$G=F_t+F\cdot F+F\cdot E-E\cdot F-2\mu\beta\|\delta_{a3}\delta_{b1}\|,\ E=\|\mathbf{e}_t^a\cdot\mathbf{e}^b\|$$

$$\text{c) }u^1=x_1\left(\frac{1}{r}\Phi_r-\frac{\varkappa_t}{2\varkappa}\right)+x_2\left(\frac{\varkappa}{r^2}\Phi_\omega-\frac{\eta}{r^2}-\frac{1}{2}\varkappa\right)$$

$$u^2=x_2\left(\frac{1}{r}\Phi_r-\frac{\varkappa_t}{2\varkappa}\right)-x_1\left(\frac{\varkappa}{r^2}\Phi_\omega-\frac{\eta}{r^2}-\frac{1}{2}\varkappa\right)$$

$$u^3=\Phi_\omega-\frac{1}{2}r^2+\frac{\varkappa_t}{\varkappa}x_3 \tag{2.4}$$

$$p=-\Phi_\tau+(\varkappa_t\Phi_\omega-\eta_t)\operatorname{arctg}\frac{x_2}{x_1}-\left(\frac{\varkappa_t}{\varkappa}\right)_t\left(\frac{1}{2}x_3^2-\frac{1}{4}r^2\right)+\Psi-\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+$$
$$+\frac{1}{8}r^4-\frac{\varkappa_t}{2\varkappa}r^2x_3+\eta\varkappa\ln r+\frac{1}{4}(\varkappa r)^2-2\nu x_3$$

where $\tau=t$, $r=(x_1^2+x_2^2)^{1/2}$, $\omega=x^3-\varkappa\operatorname{arctg}(x_2/x_1)$, η and $\varkappa$ are smooth functions of the variable t, $\varkappa_t\varkappa^{-1}:=0$ if $\varkappa=0$ and the functions $\Phi=\Phi(\tau,r,\omega)$ and $\Psi=\Psi(\tau,r,\omega)$ satisfy the system of equations

$$r\Phi_r=\Psi_\omega,\ ((\varkappa(\tau))^2r^{-2}+1)r\Phi_\omega=-\Psi_r$$

2. $H = 0, \mathbf{k} \neq \mathbf{0}$

$$\mathbf{u} = \Phi_{y_1}\mathbf{e}^1 + \Phi_{y_2}\mathbf{e}^2 + |\mathbf{k}|^{-2}(\mathbf{k}\cdot\mathbf{x})\mathbf{k}_t + |\mathbf{k}|^{-2}(\mathbf{k}_t\cdot\mathbf{x})\mathbf{k} - \\ -\frac{1}{2}|\mathbf{k}|^{-4}(\mathbf{k}_t\cdot\mathbf{k})(\mathbf{k}\cdot\mathbf{x})\mathbf{k} - \frac{1}{2}|\mathbf{k}|^{-2}(\mathbf{k}_t\cdot\mathbf{k})\mathbf{x} + \frac{1}{2}\mathbf{k}\times\mathbf{x} \tag{2.5}$$

$$p = -\varphi_t - \frac{1}{2}\mathbf{u}\cdot\mathbf{u} - |\mathbf{k}|\Psi + \frac{1}{4}|\mathbf{k}\times\mathbf{x}|^2 + \frac{1}{2}|\mathbf{k}|^{-2}(\mathbf{k}\cdot\mathbf{x})(\mathbf{k}_t\times\mathbf{k},\mathbf{x})$$

where $\tau = t$, $y_i = \mathbf{e}^i \cdot \mathbf{x}$, $\mathbf{e}^i = \mathbf{e}^i(t)$ $(i = 1, 2)$ are smooth vector functions for which the conditions $|\mathbf{e}^i| = 1$, $\mathbf{e}^1 \cdot \mathbf{e}^2 = 0$, $\mathbf{e}^i \cdot \mathbf{k} = 0$ are satisfied and the vectors $\mathbf{e}^1$, $\mathbf{e}^2$ and $\mathbf{k}$ form a right triplet

$$\varphi = \Phi + |\mathbf{k}|^{-2}(\mathbf{k}_t\cdot\mathbf{x})(\mathbf{k}\cdot\mathbf{x}) - \frac{1}{4}|\mathbf{k}|^{-4}(\mathbf{k}_t\cdot\mathbf{k})(|\mathbf{k}|^2|\mathbf{x}|^2 + (\mathbf{k}\cdot\mathbf{x})^2) \tag{2.6}$$

and the functions $\Phi = \Phi(\tau, y_1, y_2)$ and $\Psi = \Psi(\tau, y_1, y_2)$ satisfy the Cauchy–Riemann system $\Phi_{y_1} = \Psi_{y_2}$, $\Phi_{y_2} = -\Psi_{y_1}$.

3. $H = 0, \mathbf{k} = \mathbf{0}$ (potential flows)

$$\mathbf{u} = \nabla\varphi,\ p = -\varphi_t - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi$$

where $\varphi = \varphi(t, \mathbf{x})$ is an arbitrary harmonic function.

Remark 4. All of the non-potential solutions of Eqs (1.1) which have been mentioned in the theorem, are Lie solutions, that is, each of them is invariant with respect to a certain subalgebra of the maximum algebra in the Lie sense of the invariance of Eqs (1.1). For instance, solutions (2.1), (2.4) and (2.5) are invariant with respect to the one-dimensional algebras

$$\langle R(0, 0, \exp(-2\textstyle\int\zeta^1 dt)) - Z(2\nu\lambda\exp(-2\int\zeta^1 dt))\rangle$$

$$\langle J_{12} + R(0, 0, \varkappa) - Z(\eta_t + 2\nu\varkappa)\rangle \text{ and } \langle R(\mathbf{k})\rangle$$

respectively and solution (2.2) is invariant with respect to the two-dimensional algebra

$$\langle R(\mathbf{m}^1) + Z(\chi^1), R(\mathbf{m}^2) + Z(\chi^2)\rangle$$

where $\mathbf{m}^i = \alpha^{1i}\mathbf{e}^2 + \alpha^{2i}\mathbf{e}^3$, $\chi^i = -2\mu\alpha^{2i}$ $(i = 1, 2)$ and $(\alpha^{1i}, \alpha^{2i})$ $(i = 1, 2)$ are linearly independent solutions of the system of ordinary differential equations

$$\alpha_t^1 = \left(\frac{\mu_t}{2\mu} + \zeta\right)\alpha^1 - 2(\mathbf{e}_t^3\cdot\mathbf{e}^2)\alpha^2,\ \alpha_t^2 = \varkappa^1\alpha^1 - 2\zeta\alpha^2$$

The Lie solutions of the Navier–Stokes equations have been investigated earlier [6].

3. AUXILIARY ASSERTIONS

The following assertions are used in the proof of the theorem.

Lemma 1. The action of an arbitrary linear operator in the space $\mathbf{R}^3$ can be represented in the form

$$H\mathbf{x} = (\mathbf{m}\cdot\mathbf{x})\mathbf{m} - (\mathbf{n}\cdot\mathbf{x})\mathbf{n} + \gamma\mathbf{x} + \mathbf{l}\times\mathbf{x} \tag{3.1}$$

where $\gamma \in R$, $\mathbf{m}, \mathbf{n}, \mathbf{l} \in R^3$, $\mathbf{m}\cdot\mathbf{n} = 0$ and H is the matrix of the operator. In representation (3.1), the number γ and the vector $\mathbf{l}$ are uniquely defined, and the vectors $\mathbf{m}$ and $\mathbf{n}$ are defined apart from the factor of ± 1.

Proof. Suppose H^T is the transpose of H. If (3.1) is satisfied, then

$$(\mathbf{m}\cdot\mathbf{x})\mathbf{m} - (\mathbf{n}\cdot\mathbf{x})\mathbf{n} + \gamma\mathbf{x} = S\mathbf{x},\ S := \frac{1}{2}(H + H^T) \\ \mathbf{l} = \tfrac{1}{2}(H^{32} - H^{23},\ H^{13} - H^{31},\ H^{21} - H^{12})^T \tag{3.2}$$

Since the matrix S is symmetric, it can be reduced to diagonal form by orthogonal transforms, that is, an orthogonal matrix O exists such that

$$OSO^T = \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\}, \gamma_1 \geq \gamma_2 \geq \gamma_3$$

Then, the vectors $\mathbf{m}$, $\mathbf{n}$ and the number γ, which are defined by the formulae

$$\mathbf{m} = \pm O^T(\sqrt{\gamma_1 - \gamma_2}, 0, 0)^T, \ \mathbf{n} = \pm O^T(0, 0, \sqrt{\gamma_2 - \gamma_3})^T, \ \gamma = \gamma_2$$

(and only these) satisfy the requirement of the lemma.

Lemma 2. If $H^{ab} \in C^1((t_0, t_1), R)$, then $\mathbf{l} \in C^1((t_0, t_1) R^3)$ in Lemma 1, and m^a, n^a, γ are continuously differentiable functions in the set of those values of t for which $\mathbf{m}(t) \neq \mathbf{0}$ and $\mathbf{n}(t) \neq \mathbf{0}$.

Proof. The assertion of the lemma regarding the vector-function $\mathbf{l}$ is obvious by virtue of the second relation of (3.2) and, in the case of the functions m^a, n^a and γ, it is a consequence of the implicit function theorem. Actually, these functions satisfy the equations

$$\begin{aligned} &m^a n^a = 0, \ (m^b)^2 - (n^b)^2 + \gamma = H^{bb} \\ &m^a m^b - n^a n^b = \frac{1}{2}(H^{ab} + H^{ba}), \ a < b \end{aligned} \tag{3.3}$$

(Summation over the index b is not carried out here). The determinant of the derivatives of the left-hand sides of Eqs (3.3) with respect to m^a, n^a and γ has the form $-4(|\mathbf{m}|^2 + |\mathbf{n}|^2)|\mathbf{m} \times \mathbf{n}|^2$ and is therefore non-zero when $\mathbf{m} \times \mathbf{n} \neq \mathbf{0}$. By virtue of the orthogonality of the vectors $\mathbf{m}$ and $\mathbf{n}$, it is sufficient for this that $\mathbf{m} \neq \mathbf{0}$ and $\mathbf{n} \neq \mathbf{0}$.

4. PROOF OF THE THEOREM

We shall use the representation of the matrix H in the form of (3.1). The relations $\mathbf{m} \cdot \mathbf{m} = \mathbf{n} \cdot \mathbf{n}$ and $\gamma = 0$ are a necessary condition for the compatibility of equations (1.1) and (1.2).

We now consider the possible cases.

A. $\mathbf{m} \neq \mathbf{0}$, $\mathbf{n} \neq \mathbf{0}$. Then $\mathbf{l} = \sigma \mathbf{m} \times \mathbf{n}$, where $\sigma = \sigma(t)$. We introduce the notation

$$\mu := \mathbf{m} \cdot \mathbf{m} = \mathbf{n} \cdot \mathbf{n}, \ z_1 := \mathbf{m} \cdot \mathbf{x}, \ z_2 := \mathbf{n} \cdot \mathbf{x}, \ z_3 := (\mathbf{m} \times \mathbf{n}, \mathbf{x})$$

From the overdetermined system, which is obtained by substituting (1.3) into Eqs (1.1), we find the following expression for the function φ

$$\begin{aligned} \varphi = &\frac{1}{3\mu} z_1 z_2 z_3 - \frac{\sigma}{6\mu}(z_1^2 + z_2^2) z_3 + \frac{1}{2}\mu^{-3}(\mathbf{l}_t, \mathbf{m} \times \mathbf{n}) z_1 z_2 - (\mu^{-3}\mathbf{l}_t \cdot \mathbf{m} - \frac{1}{2}\mu^{-2}\mathbf{m} \cdot \mathbf{k}) z_2 z_3 - \\ &-(\mu^{-3}\mathbf{l}_t \cdot \mathbf{n} + \frac{1}{2}\mu^{-2}\mathbf{n} \cdot \mathbf{k}) z_1 z_3 + \frac{1}{2\mu}(\zeta^1 + \zeta^2) z_1^2 + \frac{1}{2\mu}(\zeta^1 - \zeta^2) z_2^2 - \mu^{-2}\zeta^1 z_3^2 + \eta^a z_a + \eta^0 \end{aligned}$$

where ζ^1, ζ^2, η^a, η^0 are smooth functions of the variable t.

If $\sigma = \pm 1$, then $\mathbf{m} \times \mathbf{n} = \mu \mathbf{e}$, $\mathbf{l} = \lambda \mathbf{e}$, where $\mathbf{e} = \mathrm{const}$, $|\mathbf{e}| = 1$, $\lambda := \sigma\mu$ and, therefore, $\lambda \neq \mu$, $\mu\mu_t = \lambda\lambda_t$. By virtue of Remark 3, it can be assumed that the relations

$$\mathbf{e} = (0, 0, 1), \ \mathbf{m} \cdot \mathbf{k} = \mathbf{n} \cdot \mathbf{k} = 0, \ \eta^3 = 0$$

are additionally satisfied.

Then, $\mathbf{k} = \varkappa \mathbf{e}$, $\eta^1 = \eta^2 = 0$ and

$$\mathbf{m} = \sqrt{\mu}\left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, 0\right)^T, \ \mathbf{n} = \sqrt{\mu}\left(\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, 0\right)^T$$

where θ and $\varkappa$ are smooth functions of the variable t for which

$$\theta_t = 2\lambda\mu^{-1}\zeta^2 - \varkappa, \ \varkappa_t + 2\zeta^1 \varkappa = 0$$

On making all of the substitutions and integrating Eq. (1.1) with respect to the function p, we obtain solution (2.1).

Since, according to Lemma 1, the vectors $\mathbf{m}$ and $\mathbf{n}$ are determined, apart from the factor ±1, then, in the case when $\sigma = \pm1$, we may put $\sigma = -1$. The equivalence transforms enable one to satisfy the following further conditions

$$\mathbf{m}\cdot\mathbf{k} = \mathbf{n}\cdot\mathbf{k},\ \eta^1 = \eta^2,\ \eta^3 = 0$$

If the notation

$$\beta := \eta^1\sqrt{2\mu},\ \mathbf{e}^1 := \frac{\mathbf{m}+\mathbf{n}}{\sqrt{2\mu}},\ \mathbf{e}^2 := \frac{\mathbf{m}-\mathbf{n}}{\sqrt{2\mu}},\ \mathbf{e}^3 := -\mu\mathbf{m}\times\mathbf{n}$$

is introduced, then, in a similar manner to the preceding case, we obtain the solution (2.2).

B. $\mathbf{m} = \mathbf{n} = \mathbf{0}$, $\mathbf{l} \neq \mathbf{0}$. In this case $\mathbf{l}_t = \mathbf{0}$, that is $\mathbf{l} = \text{const}$. Using rotations and scale transforms, we reduce $\mathbf{l}$ to the vector $(0, 0, 1)$. By virtue of Remark 3, it can also be assumed that $\mathbf{l}\cdot\mathbf{k} = \mathbf{0}$ and therefore $\mathbf{k} = (0, 0, \varkappa(t))^T$. We now integrate the overdetermined system, obtained by substituting (1.3) into (1.1), with respect to the function φ. We obtain

$$\varphi = \Phi(\tau, r, \omega) - \frac{1}{6}x_3r^2 + \frac{\varkappa_t(t)}{\varkappa(t)}\left(\frac{1}{2}x_3^2 - \frac{1}{4}r^2\right) + \eta(t)\operatorname{arctg}\frac{x_2}{x_1}$$

The variables τ, r and ω are determined as in (2.4) and the function Φ satisfies the equation

$$\Phi_{rr} + \frac{1}{r}\Phi_r + \left(\left(\frac{\varkappa(\tau)}{r}\right)^2 + 1\right)\Phi_{\omega\omega} = 0$$

On substituting the expression for φ into (1.3) and integrating the first equation of (1.1) with respect to the function p, we obtain the solution (2.4).

C. $H = 0$, $\mathbf{k} \neq \mathbf{0}$. In this case, the expression for the function φ is reduced by means of equivalence transforms to expression (2.6). Consequently, the corresponding solution of Eqs (1.1) has the form (2.5).

D. The case when $H = 0$, $\mathbf{k} = \mathbf{0}$ is obvious.

The proof of the theorem has been completed.

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